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ABSTRACT

Several characterizations of simple closed curves in two-dimensional arrays (digital pictures) are shown to be inadequate to characterize simple closed surfaces in three-dimensional arrays; but the 2d analog of a recently given 3d definition of "surface" does characterize curves.

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1. Introduction

In [1,2], a simple closed curve in a digital picture was defined as a sequence of picture points $(i_1,j_1),\ldots,(i_n,j_n)$ such that (i_r,j_r) is a neighbor of (i_s,j_s) iff. $r=s\pm 1$ modulo n. In other words, a curve is a cyclic sequence each point of which is a neighbor of its predecessor (and the first point is a neighbor of the last), but in which no other pairs of points are neighbors, so that the curve does not touch itself (or cross itself). Some other characterizations and properties of curves were established in [1,2]; see also [3]. A more precise definition, and a summary of these results, will be given in Section 2.1.

In [4] a definition was given for a simple closed surface in a three-dimensional digital array; it will be reviewed in Section 2.2, and it will be shown in Section 2.3 that the two-dimensional analog of this definition gives an alternative characterization of simple closed curves. Other characterizations of curves, on the other hand, do not have analogs that characterize surfaces.

Borders of objects are cyclic sequences, but they are not necessarily simple closed curves, since a border may pass through the same point twice (consider an object that has a thin "waist"). Nevertheless, some of the properties that hold for curves als hold for borders. If we regard a border as a set of pairs, each consisting of an object point ("1") and an adjacent

non-object point ("0"), we can develop a theory of sequences of such pairs ("bicurves") analogous to the theory of curves [5]. In fact, bicurves turn out to be the same as "connected components" of pairs. In Section 3 we summarize this theory and show that even when we regard borders as sets of object points, they are in fact just the "sets of 1's" of bicurves; moreover, we show that a set of points is a curve iff it is the "set of 1's" of two distinct bicurves.

In three dimensions, borders can also be defined as sets of pairs [6,7]. We show further in Section 3 that even when we regard them as sets of object points, they are just the "sets of 1's" of connected components of pairs, and that surfaces are the "sets of 1's" of two distinct components of pairs; unfortunately, the converse of this last result is not true.

2. Curves and surfaces

2.1 Curves

Let Σ be a square two-dimensional array of lattice points (i,j). Each point of Σ has four horizontal and vertical neighbors $(i\pm1,j)$ and $(i,j\pm1)$, as well as four diagonal neighbors $(i\pm1,j\pm1)$ and $(i\pm1,j\mp1)$. The former are called 4-neighbors and the latter 8-neighbors. A point (i,j) is called (4-,8-) adjacent to a subset S of Σ if (i,j) has a (4-,8-) neighbor in S. A subset S of Σ is called (4-,8-) connected if for any two points (i,j) and (h,k) in S, there exists a sequence of points $(i,j)=(i_0,j_0),(i_1,j_1),\ldots,(i_n,j_n)=(h,k)$, all in S, such that (i_r,j_r) is a neighbor of $(i_{r-1},j_{r-1}), 1\le r\le n$. From now on, when we omit the prefix 4- or 8-, we are making two statements in one.

Let S be any subset of Σ; the maximal connected subsets of S are called its connected components. We can also define connectedness and components for the complement $\bar{S}$ of S. In order for some of the results that follow to be valid, we will use opposite definitions (4- vs. 8-) for S and for $\bar{S}$; we can thus speak of neighbors, connectedness, etc. "in the S sense" or "in the $\bar{S}$ sense."

A sequence of points $(i_1,j_1),\ldots,(i_n,j_n)$ of Σ is called a simple closed curve (or "curve," for short) if

(i_r,j_r) is a neighbor of (i_s,j_s) iff $r=s\pm1$ (modulo n)

Thus a curve is connected, its successive points are neighbors,

and no two of its points are neighbors unless they are successive. This definition allows various degenerate cases--namely, a single point; two neighbor points; three mutually neighboring points (e.g., $\begin{smallmatrix}A & B\\ C & \end{smallmatrix}$), if we use the 8-definitions; and four points that form a 2-by-2 square $\left(\begin{smallmatrix}A & B\\ D & C\end{smallmatrix}\right)$, if we use the 4-definitions. We will exclude these cases from now on, although many of our results would hold even for them.

The following characterizations of curves are established in [1,2,3]:

<u>Proposition 1</u>. A set of points γ is a curve (i.e., the points can be numbered such that the resulting sequence is a curve) iff γ is connected, and every point in γ has exactly two neighbors in γ. //

<u>Theorem 2</u>. A set of points γ is a curve iff its complement $\bar{\gamma}$ has exactly two components, and every point in γ is adjacent (in the $\bar{\gamma}$ sense) to both of these components. //

Let $N_8(P)$ be the set of eight neighbors of P. A point P of S is called <u>simple</u> if $S \cap N_8(P)$ has just one component adjacent to P (in the S sense), and $\bar{S} \cap N_8(P)$ has at least one point adjacent to P (in the $\bar{S}$ sense). It is not hard to show that P is simple iff the analogous conditions hold with S and $\bar{S}$ interchanged, and it can also be shown that P is simple iff $S-\{P\}$ has the same number of components as S, and $\bar{S} \cup \{P\}$ has the same number as $\bar{S}$. One can then prove

Theorem 3. A set of points γ is a curve iff its complement $\overline{\gamma}$ has exactly two components, and no point of γ is simple. //

2.2. Surface points and surfaces

Let Σ be a cubical three-dimensional array of lattice points (i,j,k). Each point of Σ has the six 6-neighbors $(i\pm 1,j,k),(i,j\pm 1,k)$, and $(i,j,k\pm 1)$, and the 26-neighbors consisting of these together with $(i\pm 1,j\pm 1,k),(i,j\pm 1,k\pm 1)$, $(i\pm 1,j,k\pm 1)$, and $(i\pm 1,j\pm 1,k\pm 1)$, where all the signs are chosen independently. We define (6-,26-) adjacency, connectedness, and components just as in the two-dimensional case, and use opposite definitions (6- vs. 26-) for S and $\bar{S}$.

Curves can be defined in three dimensions as well as in two [8], using the analog of the definition in Section 2.1, or of Proposition 1. A more difficult task is that of defining simple closed surfaces; these are three-dimensional analogs of two-dimensional curves in the sense that they partition Σ into two components (an "inside" and an "outside"), as in Theorem 2. The definition of a curve in terms of a sequence of points does not apply to a surface, in which there is no linear ordering of the points. Proposition 1 cannot be used to define surfaces, since intuitively a point on a surface need not have a specified number of neighbors. The "only if" of Theorem 2 is true for surfaces, but the "if" is false, as shown in [4], because unlike a curve, a surface can touch itself without disconnecting its complement.

Simple points are harder to define in three dimensions than in two; $S \cap N_{26}(P)$ can have only one component adjacent to P (where $N_{26}(P)$ denotes the set of 26-neighbors of P), while

$\bar{S} \cap N_{26}(P)$ has many, or vice versa. If $S \cap N_{26}(P)$ has only one, then $S-\{P\}$ has the same number of components as S, and if $\bar{S} \cap N_{26}(P)$ has only one, then $\bar{S} \cup \{P\}$ has the same number as $\bar{S}$; but not conversely, i.e. even if changing P from S to $\bar{S}$ does not change the number of components of S or $\bar{S}$, it does not follow that $S \cap N_{26}(P)$ or $\bar{S} \cap N_{26}(P)$ has only one component (e.g., let S consist of a principal plane and a line perpendicular to it, and let P be the point where the plane and line meet). If we define P to be simple if $\bar{S} \cap N_{26}(P)$ has only one component adjacent to P, then evidently no surface point can be simple, so we have an analog of the "only if" of Theorem 3 (see [4] for the proof that the complement of a surface has exactly two components). The converse, however, is false; for example, if S is a hollow cube with the centers of two opposite faces pinched together until they coincide, then $\bar{S}$ has two components, and every point of S is adjacent to exactly two components of $\bar{S}$, but S is not a surface.

Let σ be a subset of Σ, and let $N_{27}(P)$ denote the set consisting of P together with its 26-neighbors. In [4] we defined $P \in \sigma$ to be a <u>surface point</u> if $N_{27}(P) \cap \sigma$ has just one component adjacent (in the σ sense) to P; $N_{27}(P) \cap \bar{\sigma}$ has exactly two components adjacent (in the $\bar{\sigma}$ sense) to P; and every $Q \in N_{27}(P) \cap \sigma$, adjacent to P in the σ sense, is adjacent in the $\bar{\sigma}$ sense to both of these components. We defined a surface point to be <u>orientable</u>

if $N_{125}(P) \cap \bar{\sigma}$ still has two components adjacent in the $\bar{\sigma}$ sense to P, where $N_{125}(P)$ is the set of 26-neighbors of P or of P's 26-neighbors (a total of 125 points, including P itself). Finally, we defined σ to be a (simple closed) <u>surface</u> if it is connected and all of its points are orientable surface points. It is not known whether the requirement of orientability in this definition is necessary. In Section 2.3 we show that the analogous definition in two dimensions, without the requirement of orientability, does in fact characterize curves.

2.3 Curve points and curves

Let γ be a subset of Σ (in two dimensions), and let $N_9(P)$ denote the set consisting of P together with its 8-neighbors. We call $P \in \gamma$ a <u>curve point</u> if $N_9(P) \cap \gamma$ has just one component adjacent (in the γ sense) to P; $N_9(P) \cap \overline{\gamma}$ has exactly two components adjacent (in the $\overline{\gamma}$ sense) to P; and every $Q \in N_9(P) \cap \gamma$, adjacent to P in the γ sense, is adjacent in the $\overline{\gamma}$ sense to both of these components.

<u>Theorem 4</u>. A connected set γ is a curve iff all its points are curve points.

<u>Proof</u>: "Only if" is easily verified. To prove "if", consider first the case where γ is 8-connected. If $P \in \gamma$ had two 8-neighbors in γ that were 8-neighbors of each other, they could not be 4-adjacent to two 4-components of $\overline{\gamma} \cap N_9(P)$ that were both 4-adjacent to P. Hence P's 8-neighbors in γ partition $N_9(P)$ into nonempty "runs" of neighbors that are 4-components of $\overline{\gamma} \cap N_9(P)$ and that are adjacent to P. Since there are exactly two such components, P must have exactly two 8-neighbors in γ, and since this is true for every $P \in \gamma$, γ is an 8-curve by Proposition 1. Similarly, let γ be 4-connected. If P had two 4-neighbors in γ that were 4-connected to each other in $N_9(P)-\{P\}$, they could not both be 8-adjacent two two 8-components of $\overline{\gamma} \cap N_9(P)$. Hence P's 4-neighbors in γ partition $N_9(P)$ into nonempty "runs" of neighbors that are 8-components of $\overline{\gamma} \cap N_9(P)$. Since there are

exactly two such components, P must have exactly two 4-neighbors in γ, and since this is true for every $P\in\gamma$, γ is a 4-curve by Proposition 1.//

The assumption that the neighbors of P are adjacent to the two components of $N_9(P)\cap\bar{\gamma}$ is essential. Suppose we call P a curve point if $N_9(P)\cap\gamma$ has just one component adjacent to P in the γ sense, and $N_9(P)\cap\bar{\gamma}$ has just two components adjacent to P in the $\bar{\gamma}$ sense. Every P in the two connected γ's shown below (the 1's denote the points of γ) has these properties, but they are not curves:

```
  1     1
1   1 1   1
1   1 1   1
  1     1
```

is not an 8-curve;

```
1 1 1
1   1 1 1 1
1 1 1 1   1
      1 1 1
```

is not a 4-curve.

3. Borders, curves, and surfaces

3.1. Components of pairs

Let Σ be a two- or higher-dimensional array, and let S be a subset of Σ. Two points P,Q of Σ will be called direct neighbors if every coordinate of P is the same as the corresponding coordinate of Q, with one exception in which the coordinates differ by 1. Thus in two dimensions, direct neighbors are 4-neighbors, and in three dimensions they are 6-neighbors. P and Q will be called indirect neighbors if they are distinct, and each coordinate of P differs by at most 1 from the corresponding coordinate of Q; in two dimensions, these are 8-neighbors, and in three dimensions, 26-neighbors. (In)direct adjacency and connectedness are defined analogously. As usual, we use opposite definitions for S and for $\bar{S}$.

In what follows, a pair means a directly adjacent pair of points (P,Q), with $P \in S$ and $Q \in \bar{S}$. We will call the points of S 1's and the points of $\bar{S}$ 0's. Two pairs (P,Q),(P',Q') will be called adjacent if P,Q,P',Q' (which need not all be distinct) lie in a 2 x 2 x 2...-dimensional hypercube K (i.e., in 2d, a 2 x 2 square; in 3d, a 2 x 2 x 2 cube; etc.); P and P' are connected in K in the S sense, and Q and Q' are connected in K in the $\bar{S}$ sense. (Note that for whichever of S and $\bar{S}$ we are using indirect connectedness, this last requirement is vacuous, since any two points in K are indirect neighbors, hence are trivially connected.) The reflexive, transitive closure of adjacency is

called connectedness; in other words, two pairs (P,Q), (P',Q') are called <u>connected</u> if there exists a sequence of pairs $(P,Q)=(P_0,Q_0)$, $(P_1,Q_1),\ldots,(P_r,Q_r)=(P',Q')$ such that (P_i,Q_i) is adjacent to (P_{i-1},Q_{i-1}), $1\le i\le r$. A maximal connected set of pairs is called a <u>component of pairs</u>.

3.2. Pairs and borders

Let C be a component of S, and D a component of $\bar{S}$, such that C and D are directly adjacent. (It is easily shown that if C and D are indirectly adjacent, they are also directly adjacent.) The set of pairs (P,Q) such that $P \in C$ and $Q \in D$ is called the <u>(C,D) border</u>.

<u>Theorem 5</u>. Any (C,D) border is a component of pairs.

<u>Proof</u>: In two dimensions, a standard "crack following" algorithm can be used to visit all the pairs of a (C,D) border in sequence, starting from an initial pair and moving to a succession of adjacent pairs, so that the pairs it visits are all connected; the fact that this algorithm visits every pair on the border thus implies that the border is connected. In three (or more) dimensions, the pairs cannot be linearly ordered, but one can still prove that the set of border pairs is connected; see [6]. Conversely, if (P,Q) is a pair of the (C,D) border, and (P',Q') is adjacent to (P,Q), we know from the definition of adjacency that P is connected to P' in the S sense, and Q to Q' in the $\bar{S}$ sense, so that $P' \in C$ and $Q' \in D$, making (P',Q') a pair of the (C,D) border too. Hence any pair adjacent to (or, by induction: connected to) a pair of the (C,D) border is itself a pair of the (C,D) border, so that the (C,D) border is a maximal connected set of pairs. //

The <u>set of 1's</u> of a set of pairs consists of the first terms of the pairs, together with the 1's that are directly connected

in K to the first terms of each adjacent pair of pairs, if we use direct connectedness for S. The set of 0's is defined analogously.

Proposition 6. The set of 1's of a connected set of pairs is connected in the S sense, and its set of 0's is connected in the $\bar{S}$ sense.

Proof: As already remarked in the proof of Theorem 5, if the pairs (P,Q) and (P',Q') are adjacent (or, by induction: connected), their 1's are connected in the S sense, and their 0's in the $\bar{S}$ sense. //

The set of 1's of the (C,D) border will be called the D-border of C (denoted C_D), and the set of 0's will be called the C-border of D (denoted D_C). Clearly $C_D \subseteq C$ and $D_C \subseteq D$. If we are using direct connectedness for S, then D_C is just the set of points of D that are directly adjacent to C, but C_D may consist of more than just the points of C that are directly adjacent to D (i.e., it also contains other 1's in the K's); and conversely. Note that these definitions differ slightly from those given in [3], where C_D was defined to be just the set of points of C that are directly adjacent to D, and vice versa.

Corollary 7. C_D is connected in the S sense, and D_C in the $\bar{S}$ sense. //

Theorem 8. Any component of pairs is a (C,D) border.

Proof: Let C' be the set of 1's and D' the set of 0's of the given component of pairs; by Proposition 6, these are connected. Let C be the component of S that contains C', and D the component

of $\bar{S}$ that contains D'. Evidently the (C,D) border contains the given component of pairs. But since the (C,D) border is a connected set of pairs (Theorem 5), and the component of pairs is a maximal connected set of pairs, they must be the same. //

Thus components of pairs are the same thing as (C,D) borders (Theorems 5 and 8).

<u>Corollary 9</u>. The set of 1's (0's) of any component of pairs is a C_D (D_C). //

Thus borders are the same thing as sets of 1's or 0's of components of pairs.

3.3. Pairs, curves, and surfaces

Theorem 10. In two dimensions, γ is a curve if it is the set of 1's of two components of pairs.

Proof: If γ is a curve, $\bar{\gamma}$ has exactly two components, and every point of γ is adjacent (in the $\bar{\gamma}$ sense) to both components. Let D be one of the components; then the set of pairs (P,Q) with $P \in \gamma, Q \in D$ is easily verified to be a connected component of pairs and to have γ as its set of 1's (even if we use direct connectedness for γ).

Conversely, let γ be the set of 1's of two components of pairs; thus by Corollary 9 there exist a component C of 1's and two distinct components D,D' of 0's such that $\gamma = C_D = C_{D'}$. Now C separates any two components of 0's that are adjacent to it [3]; hence any path of 0's from D to D' meets C, and since it must first meet C at a point of C_D, it meets γ, which proves that D and D' are in different components of $\bar{\gamma}$, call them E and E'. It is easily verified that if any point of γ were in the set of 1's of a third component of pairs, some neighbor of γ could not be in both of the first two sets of 1's. Hence $\bar{\gamma}$ cannot have more than two components, since γ must be adjacent to any component of $\bar{\gamma}$. Finally, every point of γ is adjacent, in the $\bar{\gamma}$ sense, to $D \subseteq E$ and $D' \subseteq E'$, since $\gamma = C_D = C_{D'}$. By Theorem 2, it follows that γ is a curve. //

The analogous result is not true for surfaces. A surface presumably is the set of 1's of two components of pairs; but conversely, in the pinched cube example of Section 2.2, S is

the set of 1's of two components of pairs, but it is not a surface. Note, however, that in this example, the point where the opposite faces meet belongs to two nonadjacent pairs that are in the same component of pairs; it may be possible to obtain a characterization of surfaces by ruling out this possibility.

3.4. Alternative definitions

The definitions of pair adjacency and of the set of 1's (0's) given in this section are compatible with the 2d definitions given in [5]; but other compatible definitions, which might be preferable in 3d, could also be given. In particular, let us define (P,Q) and (P',Q') to be adjacent if

a) P,Q,P',Q' (which need not all be distinct) lie in a $2\times 2\times\cdots$hypercube K; this implies that for some $0\le r,s\le d$ (where d is the dimension), i of the coordinates of P differ by 1 from those of P', and j of the coordinates of Q differ by 1 from those of Q'.

b) If we use direct connectedness for S, there is a sequence $P=P_0,P_1,\cdots,P_r=P'$ such that P_k is a direct neighbor of P_{k-1}, $1\le k\le r$, where each P_k is in $S\cap K$.

b') If we use direct connectedness for $\bar{S}$, there is a sequence $Q=Q_0,Q_1\cdots,Q_s=Q'$ such that Q_k is a direct neighbor of Q_{k-1}, $1\le k\le s$, where each Q_k is in $\bar{S}\cap K$.

We can then define the set of 1's of a component of pairs as the first terms of the pairs, together with the 1's belonging to the sequences satisfying (b), in the case where we use direct connectedness for S; and analogously for the set of 0's. These definitions are more complicated than the one given earlier, but they coincide in the 2d case, and the new definition seems preferable in the 3d case. For example, if the upper and lower planes of a 2x2x2 cube are $\begin{array}{cc|cc} P & P' & a & b \\ Q & Q' & c & d \end{array}$, the new definition does not force us to include any of the

1's in the $\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}$ plane in the set of 1's. Moreover, if the planes are $\begin{smallmatrix} P & Q' \\ Q & P' \end{smallmatrix}\Big|\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}$, by the new definition (P,Q) and (P',Q') are not adjacent. The results in this section all continue to hold for the new definition.

In [6], a definition of pair adjacency similar to the 2d definition of [5] is used. Along the lines of that definition, let us call (P,Q) and (P',Q') adjacent iff one of the following is true:

a) P is directly adjacent to P' and Q to Q' (so that P,Q,P',Q' form a 2x2 square)

b) P=P'; P,Q, and Q' are not collinear (so that they lie in a 2x2 square); and if we use direct connectedness for $\overline{S}$, the fourth point of that square is 0.

c) Q=Q'; P,P', and Q are not collinear (so that they lie in a 2x2 square); and if we use direct connectedness for S, the fourth point of that square is 1.

In (b-c), if we use indirect connectedness for $\overline{S}$(S), the fourth point of the square can be arbitrary. However, there seems to be problems in handling 26-connectedness when we use this definition in 3d. If C consists of two 26-neighboring points P,P' (and we use 26-connectedness for S), and D is the rest of Σ, we want the (C,D) border to be a connected set of pairs; but when we use this definition, no pairs with first terms P and P' can be adjacent(in fact, no Q can be 6-adjacent to both P and P')

4. Concluding remarks

The examples given in this paper show that, although surfaces in 3d are analogous to curves in 2d, they are much harder to characterize, because of their higher dimensionality. A subsequent report will deal with the theory of topology-preserving thinning in 3d; note that surfaces (not necessarily closed ones) should be invariant under such thinning.

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